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Transonic Singularities

Helmut SOBIECZKY*

Abstract

The model equations for transonic flow in the near-sonic regime are brought back to attention, as an example including classical flow examples as well as the challenging non-linear case of axisymmetric flow near Mach number unity. The purpose is to keep the fluid mechanic knowledge base by analytic treatment alive, to complement education of future engineers in the age of numerical simulation.

1. Introduction

Classical analytic modelling of transonic flow has contributed self-similar solutions to the near sonic partial differential equation during the past 40 years. A mathematically elegant and, especially in pre-computer time, practically useful set of particular solutions was described which shed light into many hitherto unexplained phenomena of flows near the speed of sound.

Today, with high performance computers and fast software to solve partial differential equations such analytic background is an 'endangered species' among the many numerical techniques for complex flows. For at least two reasons, however, it seems that a certain extract of analytic results will be beneficial in the future:

First, a new generation of engineers will easily make use of theoretical results using the graphic illustration software as being used for CFD. This way model solutions will complement teaching tools which already use CFD results for educational purposes, which has recently been described in this journal [1].

Second, despite impressive progress in CFD, there are still problems which require quite costly numerical attempts for simulation; in this case the existence and, even better, a handy availability of analytical models for such problems serves for test cases in the development and the adaptation of CFD codes, in order to verify such solutions with improving efficiency and accuracy.

In the following, near sonic flow (the „mathematical essence“ of transsonic flow) is mapped once more to hodograph variables, this way allowing the application of classical pre-CFD solution methods. Inviscid flow elements with interesting singular local or global behavior result from such approach: A relation of far field flow structure to specific body flow is illustrated here.

2. New graphics for old flow models

Mathematicians, physicists and engineers equally profit from their numerical solutions´ graphic visualization. In pre-computer time, analytic solutions had to be illustrated by sketches and line diagrams. Skalar fields in suitable sections of 3D space hardly were shown. It has a certain attractivity to review some more complex

*) Senior research scientist, DLR, Bunsenstr. 10
D-37073 Göttingen. e-mail: helmut.sobieczky@dlr.de

model solutions of this earlier time of transonics, to illustrate them with modern graphic tools: Some aspects of continuing interest may be kept alife as well as new solutions may be found much easier because of the availability of graphic visualization and animation.

3. Basic equations

Laplace´s equation or the wave equation modelling elliptic or hyperbolic linear potential models provide the basic solutions of low and high speed flow, respectively. Harmonic solutions for a velocity potential in plane 2D or axisymmetric physical space define a set of flow models which due to its simplicity stays important for education in fluid mechanics.
In compressible flow, a somewhat unusual unified form of the perturbation potential equation is investigated to apply familiar techniques of linear flow types to the non-linear ones:,

$$l \cdot (\gamma + 1)^k \cdot |\varphi_x|^k \cdot \varphi_{xx} - \varphi_{yy} - j\varphi_y / y = 0 \qquad (1)$$

The use of 3 switch parameters (*j*, *k*, *l*) in this equation covers the following flow types:

- Integer *j* distinguishes between plane (*j*=0) and axisymmetric (*j*=1) flow.
- Integer *k* distinguishes between perturbed linear sub- or supersonic flow (*k*=0) and perturbed sonic flow (*k*=1) with γ the ratio of specific heats, and finally
- Integer *l* distinguishes between locally subsonic (*l*= -1) and locally supersonic (*l*=1) flow.

Perturbation velocity components (u,v) and physical coordinates (x,y) may be obtained from similarity variables (X, Y, U, V) with free scaling parameters A and B

$$x = A \cdot X$$

$$y = A \cdot B^{-k/3 \cdot (1 + 2j)} \cdot (\gamma + 1)^{-k/2} \cdot Y$$

$$(2)$$

$$\frac{u}{u_{ref}} - 1 = B^{(1 + j - (1 - j) \cdot k/3)} \cdot (l^k \cdot U)^{(1 - k/3)}$$

$$y^j \cdot \frac{v}{u_{ref}} = A^j \cdot B^{(1 + j)} \cdot (\gamma + 1)^{(k \cdot (1 - j)/2)} \cdot (1 - k/3) \cdot V$$

A hodograph transformation converts equation (1) into a set of coupled Beltrami equations (3) for velocity variables (U, V) and the coordinates of physical space (X, Y), valid in a parametric „Rheograph“-plane (s,t):

$$V_t - Y^j(s, t) \cdot U_s = 0$$
$$V_s - l \cdot Y^j(s, t) \cdot U_t = 0 \qquad (3a)$$

$$X_s - U^{k/3}(s, t) \cdot Y_t = 0$$
$$X_t - l \cdot U^{k/3}(s, t) \cdot Y_s = 0 \qquad (3b)$$

4. Similarity solutions

This unified system for variables of state and geometry (U, V, X, Y) suggests a generalization of the knowledge base we have for the solution of elliptic and hyperbolic problems using real and complex characteristics

$$l = -1: \qquad \begin{aligned} \xi &= t + is \\ \eta &= t - is \end{aligned} \qquad (4a)$$

$$l = 1: \qquad \begin{aligned} \xi &= t + s \\ \eta &= t - s \end{aligned} \qquad (4b)$$

This is the method of conformal mapping for Laplace´s equation modelling the familiar case (*j, k, l* = 0, 0, -1), the method of characteristics for the wave equation (*j, k, l* = 0, 0, 1) and Stokes' potential for axisymmetric flows. For plane and axisymmetric linear (*k=0*) problems, certain parallels exist and have been treated by Weinstein [2].
Remains a combination of these parameters for near sonic flows *k*=1, i. e. flow fields with a perturbation of sonic flow, describing either a global flow or a local flow pattern surrounding a spot with sonic flow condition. The author, 3 decades ago, investigated Eq. (3) for plane flow *j*=0, but axisymmetric near sonic flow *j*=1 remained much less understood and documented with analytically exact solutions. Nevertheless, a couple of axisymmetric flow models verifying early results suggested by Guderley [4] can be found by extending the solution techniques proven useful for *j* = 0 to case studies with *j* = 1. Reviewing these and trying to keep the door open for further investigations, it seems practical to present the reduction to a coupled system of ODE´s and including a new idea for logarithmic terms which has proven fruitful in cases of complex flow singularities in the linear models explained above.

Separation of variables in polar coordinates,

$$\begin{aligned} U &= r^n \cdot (f_u(\varphi) + \ln r \cdot g_u(\varphi)) \\ V &= r^{n + j \cdot b} \cdot (f_v(\varphi) + \ln r \cdot g_v(\varphi)) \\ X &= r^{b + k \cdot n/3} \cdot (f_u(\varphi) + \ln r \cdot g_x(\varphi)) \\ Y &= r^b \cdot (f_y(\varphi) + \ln r \cdot g_y(\varphi)) \end{aligned} \qquad (5)$$

for Eq. 3, with (r,ϕ) polar coordinates in (s,t), results in a non-linear system of coupled ODE´s for f_u, ... and g_u, ... if *k*=1, and *j*=1. It can be seen easily, that for the simpler linear cases j=0 **and/or** k=0 the familiar exact harmonic solutions without logarithmic terms are resulting immediately.
Several aerodynamically interesting flow models have been found using (5), which can serve for accuracy tests of numerical algorithms. For most applications to model physically meaningful flow the logarithmic terms have not been included yet, g_u, ... = 0. With the remaining possibilities it can be shown that only the ratio

$$\nu = n/b \qquad (6)$$

is relevant for describing a variety of self-similar solutions. For near sonic flow, the complete model consists of both subsonic (U<0) and supersonic (U>0) parts with *l* = sign(U):
Fig 1 is a sketch of this parametric hodograph plane, termed “Rheograph plane” because of the validity of a rheoelectric analogy initially aiding the understanding of boundary value problems.

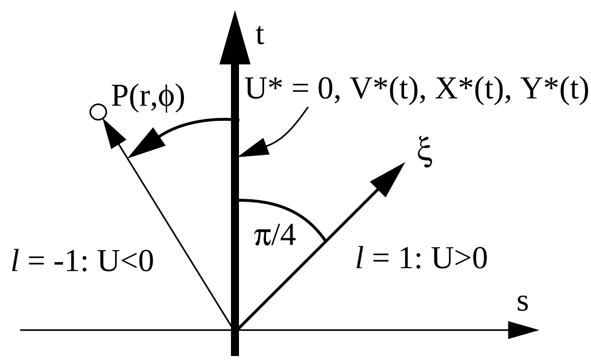

Fig. 1: Rheograph plane for near sonic flow: Polar coordinates and domain separation

Connecting both parts of the solution to represent a mixed type flow example with smooth transition from subsonic to supersonic flow is then suitably carried out by choosing a value ϕ^* from which to integrate the system for the ODE´s in different directions of the polar angle ϕ. Isofringes visualization of the results in physical variables U,V(X,Y) is carried out easily for a given solution (5): Global and local solutions for perturbed compressible flow can be discussed this way.

To illustrate the potential of this flow modelling, a well-known exact supersonic flow model is identified to have a near sonic version, which was never found by other transonic methods, and finally we discuss the definition of body flows in sonic free-stream conditions.

5. Conical flow and the near sonic equation

As it is well-known from the literature, Taylor and Maccoll [5] have described supersonic flow past a cone with attached bow wave 20 years before the family of near sonic self similar solutions [4] has been presented, then with only a few practical applications. Supporting conical behavior, parameter $\nu = n/b$ must be zero, with n=0 and b=1 to obtain a welcome proportionality between the independent Rheograph variables (s,t) and the resulting physical coordinates of the meridional plane (X,Y). This solution was presented in [6]. For another investigation we used an inverse method of characteristics [7] for the full axisymmetric Euler equations to design a supersonic cone flow. An extension of the solution toward the axis within the cone body was found where the solution ran into a limit cone turning back onto itself, forming a second lobe of the flow model within the same physical space. This way the axis cannot be reached which is consistent with the familiar example that a 2D supersonic source/sink model does not include the origin. The near sonic conical solution results in the same phenomenon by a singular behavior of the solutions f_u, f_v, $f_x(\phi)$ before a zero of $f_y(\phi)$, indicating the axis Y=0, is reached.

We consider it an advantage of the near sonic equation in its shape (3) to have yielded conical flow so easily, while the pioneering work dealing with (1) after the Taylor-Maccoll solution being already well-known, did not identify its near sonic version.

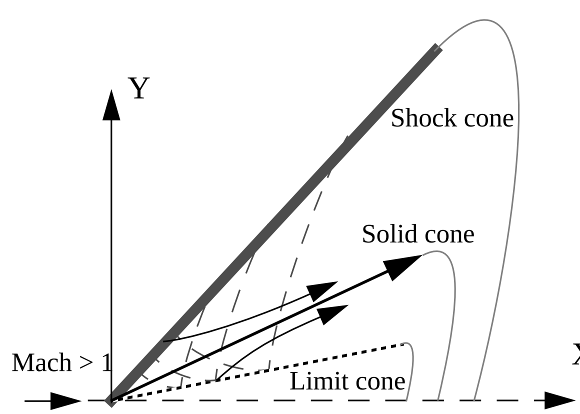

Fig. 2: Conical flow: Analytical continuation of the flow model within the solid cone yields a limit cone.

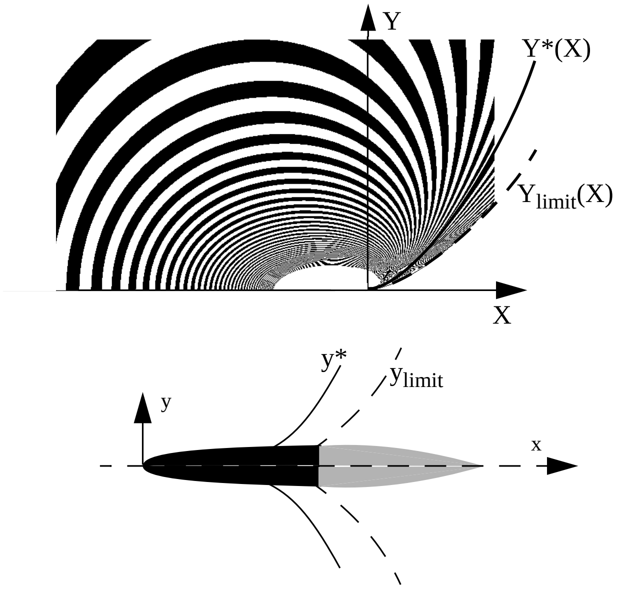

Fig. 3: Guderley´s sonic dipole flow: Isobar visualization; sonic line $y^*(x) \sim x^{7/4}$ and limit line $y_{limit}(x) \sim x^{7/4}$, consistent with front body radius $y_{body} \sim x^{1/4}$ and arbitrary continuation to obtain a closed body.

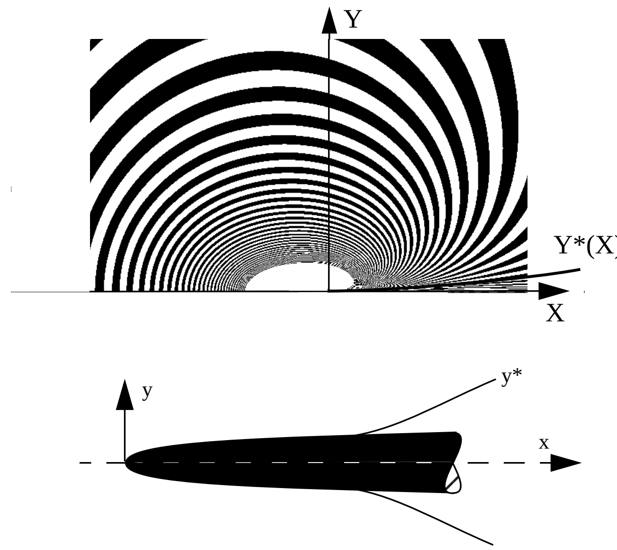

Fig. 4: Half body radius $y_{body} \sim x^{2/5}$ of infinite length in sonic free stream conditions. Sonic line $y^*(x) \sim x^{8/5}$, no limit line.

6. Far field singularities and body flows

One of the most challenging goals of analyzing solutions to the axisymmetric near sonic equation in its shape (1) was the finding of the far field behavior of a body in sonic flow $M_{inf} = 1$. Following the flow topology analysis of Guderley [4] the ratio $\nu = n/b = -9/7$, again without needing the logarithmic terms, yields this solution.

For plane 2D flow (j = 0) the field is found with $\nu = n/b = -5/4$, the values result from the requirements to represent a body with finite length in sonic flow (Fig. 3). This is achieved by obtaining a flow with a regular behavior at the limit characteristic (ξ-axis in Fig. 1) providing the freedom to continue and closing the body downstream of this limit. Parameters $\nu > -9/7$ result in body flows which do not permit a closing: all characteristics along the body in the supersonic domain reach the sonic line and will therefore influence the subsonic domain, the solution is valid only for the infinitely long half body: Fig. 4 shows body and flow field for $\nu = -9/8$.

The general relation between sonic line and body contour is found to be

$$y_{body} = x^{\alpha}$$

$$y_{M=1} = x^{\sigma} \qquad (7)$$

$$\sigma = 2 - \alpha$$

whithin a range of $1/4 < \alpha < 1$.

7. Other results

A bewildering variety of solutions to the system of ODE´s resulting from the quasi-harmonic ansatz (5) can be found, challenging an interpretation as flow models with special topology. For $k=0$ the type parameters j and l include all harmonic potential solutions for Laplace´s and wave equation. The logarithmic extension allows for more complex boundary value problems like the problem of the normal shock on the curved wall as suggested by Gadd [8]. There seem still many more possibilities to use these models for analytical description of local and global flows.

For linear equations (3), (j,k=0,1 and j,k=1,0), superimposition of particular solutions is the method for constructing flows with arbitrary boundary values. For the non-linear system modelling axisymmetric near sonic flow, numerical techniques based on iterative updating suitably linearized equations, guided by the exact linear systems are suggested but experience with this approach is small.

8. Conclusion

An effort is made to keep small disturbance theory alive in the age of numerical simulation of complex flow problems: A unified approach is pointed out to illustrate relationship between very familiar solutions to potential theory and the relatively poorly investigated near sonic axisymmetric flows, which do not result in linear equations by transformation to hodograph variables. Nevertheless the mapping of axisymmetric flow to a hodograph plane is illustrative and carries many results hitherto unexploited. The problem of creating body flows in Mach number unity has been chosen here to illustrate one of many challenges to find and to interpret solutions to the non-linear equations.

Education of a future generation of aerospace engineers will need model problems to support the development of creativity in design with a solid mathematical background trained in case studies like the ones briefly illustrated here.

9. References

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