2. BELTRAMI DIFFERENTIAL EQUATION IN PLANE POTENTIAL THEORY

2.1 Potential flow in 2D physical space

We consider steady, two-dimensional, isentropic and irrotational flow of a polytropic, inviscid gas. The basic equations of motion are then determined by

$$div(\rho, \vec{v}) = 0 \tag{1 a}$$

$$curl(\vec{v}) = 0 \tag{1 b}$$

the continuity equation and irrotationality, respectively, with ρ the density and $\vec{v}$ the velocity vector in physical space. Isentropic gas properties determine velocity q, sonic speed a and density ρ as functions of the Mach number M, for given stagnation conditions, denoted here with subscript 0:

$$\begin{aligned} q &= |\vec{v}| = aM \\ a^2 &= a_0^2 - \frac{\gamma-1}{2}q^2 \\ \frac{\rho}{\rho_0} &= \left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{1}{\gamma-1}} = F(M) \end{aligned} \tag{2}$$

With (1a, 1b) we may define a velocity potential Φ and a stream function Ψ, with their gradients in the two directions x, y of 2D physical space equal to the velocity components u, v in these directions:

$$\begin{aligned} \phi_x &= \frac{\rho_0}{\rho}\Psi_y = u = q\cos\vartheta \\ \phi_y &= -\frac{\rho_0}{\rho}\Psi_x = v = q\sin\vartheta \end{aligned} \tag{3}$$

where ϑ is the flow angle. The system (3) is a generalization of the Cauchy-Riemann equations, so-called ***Beltrami*** equations. Elimination of Ψ or Φ yields Poisson equations for Φ or Ψ, respectively:

$$\phi_{xx} + \phi_{yy} = -\frac{\rho_x}{\rho}\phi_x - \frac{\rho_y}{\rho}\phi_y \tag{4 a}$$

$$\Psi_{xx} + \Psi_{yy} = \frac{\rho_x}{\rho}\Psi_x + \frac{\rho_y}{\rho}\Psi_y \tag{4 b}$$

with ρ a function (2) of M, and therefore

$$\rho = \rho(q/a) = \rho(\phi_x^2 + \phi_y^2) \qquad (5)$$

the system (3) and the equations (4) are nonlinear. Furthermore, the system is of elliptic type if M < 1, and of hyperbolic type if M > 1, with a parabolic type dividing line M = 1, the sonic line.

2.2 Potential flow in the Hodograph plane

The aforementioned nonlinearity of the basic system (3) may be avoided if a new pair of independent variables is introduced to replace physical coordinates x, y. These variables are suitable functions of the velocity components, they are called hodograph variables. A special pair of such variables is consisting of the flow angle, ϑ and a function of the Mach number, known also as the Prandtl-Meyer turning angle

$$\upsilon = \int_{a^*}^{q} \sqrt{|M^2 - 1|}\frac{dq}{q} \qquad (6)$$

with a* defining the critical velocity.

The coefficient

$$K = K(M(\upsilon)) = \frac{\rho_0}{\rho}\sqrt{|M^2 - 1|} \qquad (7)$$

will also be used in the following system. The new variables υ, ϑ may either be used directly to define a hodograph plane wherein the basic Beltrami system becomes linear:

$$\begin{aligned} \phi_\upsilon &= K(\upsilon)\Psi_\vartheta \qquad (\upsilon \ge 0, M \ge 1) \\ \phi_\upsilon &= -K(\upsilon)\Psi_\vartheta \qquad (\upsilon \le 0, M \le 1) \\ \phi_\vartheta &= K(\upsilon)\Psi_\upsilon \end{aligned} \qquad (8)$$

or υ and ϑ more generally are functions of a computational working plane obtained from the basic υ, ϑ hodograph by conformal (for $M \le 1$) or characteristic (for $M \ge 1$) mapping. For subsonic flow including sonic conditions, ($M < 1$, $\upsilon < 0$), conformal mapping defines a working plane

ζ,

$$\zeta_0 = \upsilon + i\vartheta \tag{9 a}$$

$$\zeta = s + it = E(\zeta_0) \tag{9 b}$$

with the mapping function E. The basic system in ζ becomes then

$$\begin{aligned} \phi_s &= -K(\upsilon(s,t))\Psi_t \\ \phi_t &= K(\upsilon(s,t))\Psi_s \end{aligned} \tag{10}$$

with

$$\begin{aligned} \upsilon_{(s,t)} &= Re(E^{-1}(\zeta)) \\ \vartheta_{(s,t)} &= Im(E^{-1}(\zeta)) \end{aligned} \tag{11}$$

Equations (10) form, as (8), a linear Beltrami system, while (3) is nonlinear. Elimination of Ψ or Φ yields linear Poisson equations for Φ or Ψ, respectively:

$$\phi_{ss} + \phi_{tt} = \frac{K_s}{K}\phi_s + \frac{K_t}{K}\phi_t \tag{12 a}$$

$$\Psi_{ss} + \Psi_{tt} = -\frac{K_s}{K}\Psi_s - \frac{K_t}{K}\Psi_t \tag{12 b}$$

As we will see later, boundary value problems for practically interesting solutions of the basic system (8) may be significantly simpler to solve in a working plane ζ with (10) rather than in the original ζ_0 where (8) is valid.

The same is true, in principle, for the supersonic part of the flow. Here we introduce characteristic variables with a suitable mapping function H,

$$\begin{aligned} \xi &= H(\vartheta + \upsilon) \\ \eta &= H(\vartheta - \upsilon) \end{aligned} \tag{13}$$

yielding the system valid in the ξ, η plane

$$\begin{aligned} \phi_\xi &= K(\upsilon(\xi,\eta))\Psi_\xi \\ \phi_\eta &= -K(\upsilon(\xi,n))\Psi_\eta \end{aligned} \tag{14}$$

or equivalently,

$$\left.\frac{d\Psi}{d\phi}\right|_{\xi,\eta = const} = \pm K^{-1} \tag{15}$$

which is the basic relation for the method of characteristics to integrate the flow equations (8) for M > 1.

2.3 Near sonic flow in the hodograph plane

A given solution of (8) allows the integration of physical coordinates x, y with the formula

$$dz = dx + idy = e^{i\vartheta}\left(d\phi + i\frac{\rho_0}{\rho}d\Psi\right)/q \tag{16}$$

For flows with only small perturbations to a sonic parallel flow,

$$\begin{gathered}(M-1) \ll 1 \\ \vartheta \ll \pi/2\end{gathered} \tag{17}$$

we may eliminate Φ and Ψ so that a basic system for the physical plane coordinates x, y is obtained. Furthermore, introduction of a similarity parameter σ allows the use of reduced variables for place (x, y) and state (q, ϑ) which contain the well-known Transonic similarity laws

$$\begin{gathered}S = \pm 2\cdot 3^{-1}\sigma^{-1}(\gamma+1)^{1/2}a\left|1-\frac{q}{a^*}\right|^{3/2} \\ T = \sigma^{-1}\cdot\vartheta \\ X = \phi/a^* \\ Y = \sigma^{1/3}\cdot 3^{1/3}[2^{-1}(\gamma+1)]^{\frac{1}{\gamma-1}+\frac{1}{3}}\cdot\Psi/a^*\end{gathered} \tag{18}$$

with positive S for $q \ge a^*$ and negative S for $q \le a^*$, thus S= 0 equivalent to sonic flow conditions.

The basic system (8) then yields a corresponding Beltrami system for the reduced physical plane Variables X, Y in the reduced variables of state working plane S, T:

$$\begin{aligned}X_S &= \left|S^{1/3}\right|Y_T \qquad (S\ge 0,\ M\ge 1) \\ X_S &= -\left|S^{1/3}\right|Y_T \qquad (S\le 0,\ M\le 1) \\ X_T &= \left|S^{1/3}\right|Y_S\end{aligned} \tag{19}$$

Linearity again, and the simple structure of the coefficient gave rise to extensive studies of this system and the structure of its solutions. It is equivalent to the well known Tricomi equation for near sonic flow[3], Also, it is a special case of Generatized Axisymmetric Potential Theory[4]. Numerous particular solutions were described[5] and used for better understanding of experimentally observed transonic flow phenomena at a time, when computers and numerical methods were still not available. An analytical example for transonic airfoil flow will illustrate the possibilities of this approach.

2.4 Electric Potential in a Plane Conductor

Let us consider the distribution of electric current in a three-dimensional conductor. Let E be the electrical potential and λ (x, y, z) be the conductivity. The current intensity, di, which crosses a surface element, dS is given by Ohm's law:

$$di = -\lambda \frac{dE}{dn} dS \qquad (20)$$

where n is the surface normal to dS. In the case of a two-dimensional (x, y) conductor, variable conductivity can be simulated by constant conductivity but variable thickness distribution of the conductor, h(x,y). The current density, di, crossing the surface element, dS, described by the perpendiculars along the arc, ds, in the x, y plane, is

$$di = -\lambda \cdot h(x, y) \frac{dE}{dn} ds \qquad (21)$$

With the assumption of conservation within the conductor,

$$div(hgradE) = 0 \qquad (22)$$

a partial differential equation is obtained then for E:

$$E_{xx} + E_{yy} = -\frac{h_x}{h} E_x - \frac{h_y}{h} E_y \qquad (23)$$

There exists, moreover, a current function, W, which is associated to the electrical potential by the Beltrami system

$$E_x = \frac{1}{\lambda_h} W_y \qquad E_y = -\frac{1}{\lambda_h} W_x \qquad (24)$$

Having described flows by different forms of Beltrami equations earlier, we note here the analogy between subsonic gas flow and electric current variables: there are obviously two types of analogy[6,7], called Rheoelectric Analogies A and B:

<u>Analogy A</u>

$$\phi \equiv E \qquad \Psi \equiv W \qquad \lambda h \equiv \begin{cases} \rho/\rho_0 & Equ(3) \\ K^{-1} & Equ(8), (10) \end{cases} \qquad (25)$$

Analogy B

$$\phi \equiv W$$
$$\Psi \equiv E$$
$$\lambda h \equiv \begin{cases} (\rho/\rho_0)^{-1} & Equ(3) \\ K & Equ(8), (10) \end{cases} \qquad (26)$$

As we stated earlier, the existence of these analogies led to many applications, mainly to solve system (3) for complicated flow boundary conditions and most effectively for the incompressible limit $\rho = \rho_0$ at a time when computers where not operational or available. From an experimental standpoint, the simpler operation is the measurement of the electrical potential, E. Analogy A thus gives, with measured electrical potential, a distribution of Φ in the analog working plane (x, y), (υ/ϑ) or (s, t) in (3), (8) or (10), respectively, while analogy B provides a solution of the Ψ-distribution, for a given and analogously solved boundary value problem in the physical or hodograph plane.

It is the purpose of this paper to illustrate some applications of the outlined analogies to transonic flow problems, in particular airfoil design. At a time when the analogy already was used for numerous problems[8], transonic applications seemed impossible due to difficulties near the sonic flow conditions, as will be illustrated later.

The following chapter will outline a new idea, which led to fruitful use of the analogy in transonic airfoil design. At the same time, however, digital computers became widely used and at first the use of analog computation of purely elliptic (subsonic flow) problems was more or less terminated. But transonic computational aerodynamics remained a problem so that at least few researchers considered it worth to investigate the use of analog computation. Results shown in this paper stem from such research.

Finally, however, rapid progress in numerical methods - also in transonic aerodynamics invited to introduce some of the ideas developed with the analogy into digital computation and thus obtain solutions now much more economically. Results of these methods are presented here, too, and it is the purpose of this paper to present a recent effective numerical approach to transonic airfoil design as a logical step to be taken after some very educational experiments with rheoelectric analogy.